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13. ABSTRACT (Maximum 200 words)

In a previous paper, the author has proposed a model of general tactical C^3 processes. In that approach, an initial attempt was made at axiomatizing the essence of C^3 variables were identified, such as a node state variables, including positions, equations of motion, damage level, supply level, and states of knowledge concerning other nodes, both friendly and adversary. Other types of C^3 variables were also identified, such as detections, hypotheses formations, incoming information in the form of signals and weapons, and data fusion. Then, primitive relations involving these variables were postulated. This, in turn, leads to a basic theorem showing the recursive dynamic evolution of a typical node state complex. Because the theory established here is a formal one, both stochastic and other logical interpretations – such as fuzzy logic – can be formulated compatible with the above-mentioned theorem. All of this was shown to lead to inputs into an overall C^3 decision game between the friendly and adversary forces, where each player's move corresponds to a choice of design in terms of the primitive relations among the C^3 variables. In addition, implementation issues concerning computations involved with the above theory were considered. A new technique was exhibited which combined "exact linearizations" with gaussian sum representations of distributions resulting in general in substantial reductions of computations.

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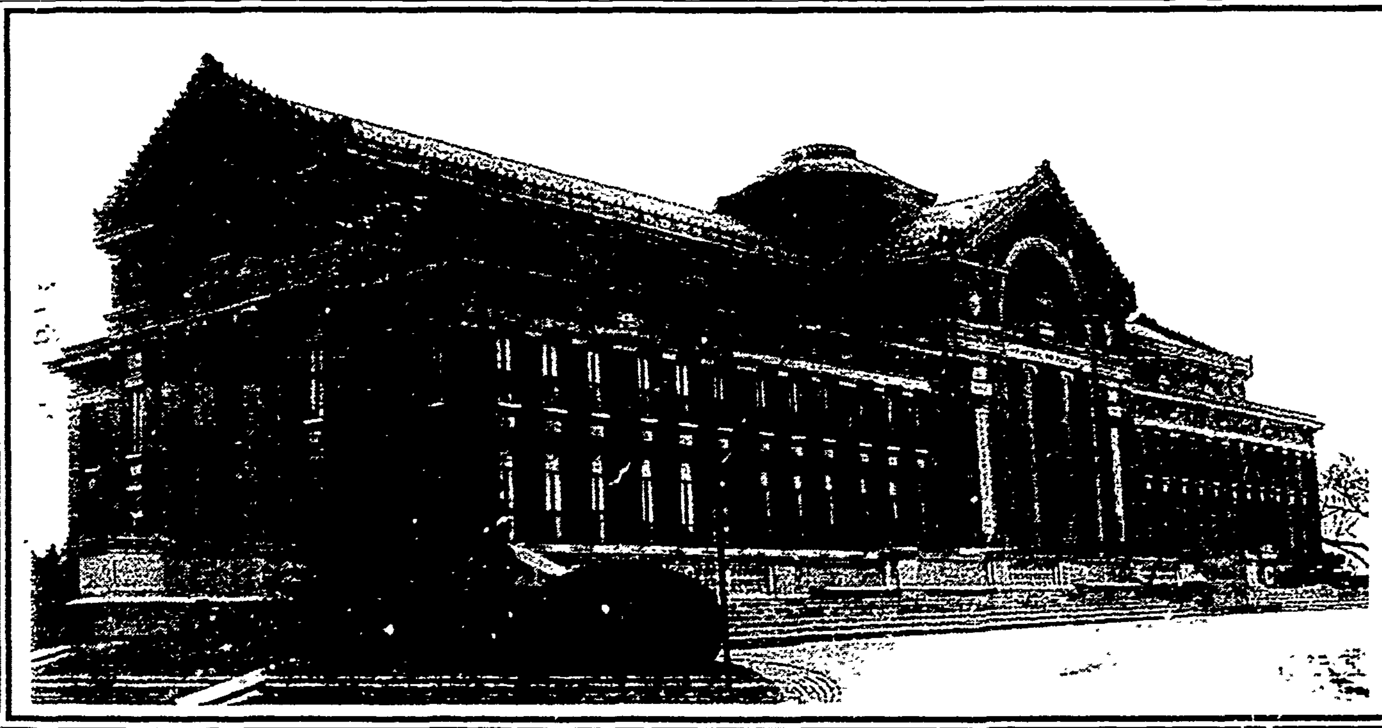

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TOWARD A GENERAL THEORY OF C^3 PROCESSES: PART 2

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ABSTRACT

In a previous paper, the author has proposed a model of general tactical C^3 processes. In that approach, an initial attempt was made at axiomatizing the essence of C^3. First, relevant C^3 variables were identified, such as node state variables, including positions, equations of motion, damage level, supply level, and states of knowledge concerning other nodes, both friendly and adversary. Other types of C^3 variables were also identified, such as detections, hypotheses formations, incoming information in the form of signals and weapons, and data fusion. Then, primitive relations involving these variables were postulated. This, in turn, leads to a basic theorem showing the recursive dynamic evolution of a typical node state complex. Because the theory established here is a formal one, both stochastic and other logical interpretations - such as fuzzy logic - can be formulated compatible with the above-mentioned theorem. All of this was shown to lead to inputs into an overall C^3 decision game between the friendly and adversary forces, where each player's move corresponds to a choice of design in terms of the primitive relations among the C^3 variables. In addition, implementation issues concerning computations involved with the above theory were considered. A new technique was exhibited which combined "exact linearizations" with gaussian sum representations of distributions resulting in general in substantial reductions of computations.

In conjunction with the above model, some basic questions have remained unanswered: How can submodel choices for the key data fusion aspect affect the overall C^3 model ? Should probability or fuzzy sets be used in implementing data fusion ? Can one compare and contrast these choices quantitatively ? In response to these issues, the current paper considers treating comparisons and contrasts of choice of uncertainty measures from a game theoretic viewpoint, extending previous work of DeFinetti and Lindley. An outline of a procedure is also presented here for directly incorporating the comparison of choices into the overall C^3 model and determining the effect upon performance and effectiveness.

1. INTRODUCTION

The author proposed a comprehensive theory of tactical C^3 systems based upon a microscopic bottoms-up viewpoint [1]. This approach is in contradistinction to the more standard global/macroscopic approaches taken by many researchers. (See [1] and [2] for brief surveys of the field; see [3] for a more extensive overview of work during the past ten years.)

Efforts are underway in relating the author's work with that of others' independent approaches, including the work of Rubin and Mayk [5],[6], Levis et al. [7],[8], Gardner [4], and Ingber [9],[10]. Future research will concentrate on developing fully connections with these researchers.

In summary, previously in [1] C^3 processes were considered as interacting networks of node complexes of decision-makers and analyzed basically as follows:

1. C^3 primary variables can be identified and classified into a basic taxonomy:

GLOBAL
- Complexity, distribitivity, hierarchy
- World views, politics

INTRANODAL (Within Nodes)
- Node state proper N
  - Number of troops
  - Threat levels
  - Equations of motion
  - Supply/attrition levels
  - Damage levels
  - Importance measures
- Knowledge aspect K
  - Algorithms available
  - Estimation of other intranodal variables, friendly and/or adversary
- Node structure T
  - Detection Det
  - Hypotheses formulations Hyp
  - Consultations Con
  - Algorithm selection Alg
  - Data Fusion DF
  - Decision processes Dec

INTERNODAL (Between nodes)
- Node output responses R
- Medium/environment parameters Q
- Node reception/input"signals" S

2. C^3 system primitive sufficiency relations can be determined, leading to a full formal theory of node state evolution:

$$
AX \begin{cases} (N^{++}|R^{++},T,N^{+},S,R^{-},\tilde{N}) = (N^{++}|R^{++},N^{+}) \\ (R^{++}|T,N^{+},S,R^{-},\tilde{N}) = (R^{++}|Dec,N^{+}) \\ (T|N^{+},S,R^{-},\tilde{N}) = (T|N^{+}) \\ (N^{+}|S,R^{-},\tilde{N}) = (N^{+}|S,N) \\ (S|R^{-},\tilde{N}) = (S|R^{-}) \\ (R^{-}|\tilde{N}) = (R^{-}|N) \end{cases} \tag{1.1}
$$

where $\tilde{N}$ is the set of all N's and their previous states and where superscript ++, +, (blank), - indicate relative node processing times.

3. Under the assumption of 2 and general (CE) conditional event algebra extending both ordinary probability logic and fuzzy logic (in Zadeh's sense), among others [11], N^{++} can be obtained recursively as an explicit functional of the primitive sufficiency relations in 2 and logical conjunction and disjunction operators in an integrated-out chaining of conditional forms. Symbolically,

$$
N^{++} = \mathcal{G}(AX{:}N{;}CE;\ \cdot\ ,\ v\ )\ ;\ \text{all } N\ , \tag{1.2}
$$

for computable functional $\mathcal{G}$. (See [1], Theorem 4.1.)

4. Under the further assumptions of a full algebraic logic description pair ALDP, i.e., compatible semantic evaluation (or models) $\| \ \|$ is added to CE, the node states' general distributions (or possibility functions, etc.) can be obtained:

$$
p_{ALDP}(N^{++}) = \mathcal{G}(p_{ALDP}(AX);p_{ALDP}(N);\| \cdot \| ,\| v \|), \tag{1.3}
$$

for all N with functional $\mathcal{G}$ computable.

5. The overall C^3 system's averaged value or measure of central tendency is determined as

$$
\bar{p}_{ALDP}(C^3) = AV\Big(p_{ALDP}(N^{++});\ N \in C^3\Big), \tag{1.4}
$$

6. C^3 functions of primary variables and their distributional and logical characteristics can be ascertained:

MOE Measures of effectiveness/system performance or specification;

Synchronicity / asynchronicity,
Timeliness/duration of battle,
Political gain/loss,
Monetary gain/loss,
Overall attrition/supply levels,
Overall damage and/or morale,

$$
MOE(C^3) = \mathcal{M}(\bar{p}_{ALDP}(C^3)) \ , \tag{1.5}
$$

for some functional $\mathcal{M}$ computable via transform techniques.

7. Health of overall C^3 system (friendly or adversary, separately) is

$$
HLTH(C^3) = \mathcal{H}(MOE(C^3)) \ , \tag{1.6}
$$

for some computable functional $\mathcal{H}$.

8. Loss of overall C^3 game (C^3_{Fr},C^3_{Ad})

$$
\mathcal{L}(HLTH(C^3_{Fr}),HLTH(C^3_{Ad}))=LOSS(C^3) \ , \tag{1.7}
$$

where $\mathcal{L}$ is a loss function, Fr is the index representing the friendly C^3 system, Ad is the index representing the adversary C^3 system, and it is assumed that each C^3 system can be identified as the tuple

$$
C^3 = (p_{ALDP}(AX),\ p_{ALDP}(N)\ ;\ ALDP,\ \| \cdot \| ,\| v \|)\ , \tag{1.8}
$$

and where equivalently one can write

$$
LOSS(C^3) = \mathcal{N}(C^3_{Fr}\ ,\ C^3_{Ad})\ , \tag{1.9}
$$

where

$$
\mathcal{N} = \mathcal{L} \circ \mathcal{H} \circ \mathcal{M} \ . \tag{1.10}
$$

9. Full C^3 Design Game $(C^3_{Fr}\ ,\ C^3_{Ad}\ ;\ LOSS)$ is thus determined through steps 1-8.

10. Obtain for C^3 Design Game, bayes decision functions, minimax, least favorable distributions, and, more generally, sensitivity of loss to changes in the designs, i.e., choices for functional descriptions for AX and ALDP, etc., for each side. (Again, see [1] for further details.)

In order to implement the above steps, each intermediate computation must be reassessed and possibly expanded and evaluated appropriately. Pruning of the more remote possibilities of combinations of C^3 variable values can be of great benefit here. Recently, P.Girard [12] has shown that a feasible and faithful implementation scheme for at least steps 1-4 relative to ALDP choice CPL (conditional probability logic) can be obtained for a simplified version of the outer/inner air battle scenario.

In [13] Goodman established a beginning of a general theory for data fusion and pointed out relations to C^3 systems as a whole. In [14], a particular implementation of data fusion was initiated via the concept of "measure-free" conditional events - alluded to in the above development of the C^3 Design Game - with emphasis on developing a full conditional probability logic. In [16], this idea was extended and modified for the possible choice of Zadeh's fuzzy sets and logic, as well as for related logics.

A basic issue for all of the above *is the actual choice of ALDP: probability? fuzzy sets? Dempster-Shafer measures?* Obviously, if all pertinant information is sensor-oriented and/or stochastic in nature with reasonably well-defined distributions available, then PL (probability logic) should be considered. On the other hand, if natural language descriptions are present in some quantity, then possibly Zadeh's fuzzy set scheme is more apropos. Other situations can arise, where the Dempster-Shafer measures appear attractive. Thus, what to do?

2. EXTENSION OF DEFINETTI-LINDLEY GAME: BASIC CONCEPTS

In response to the last-mentioned issue in the last section where one wishes to choose the most apropos ALDP for a given situation, DeFinetti [16] indeed showed the following: The class of all finitely additive probability measures coincides with the admissible class of nonrandomized decisions for a particular decision game. In that game, the aggregated loss is a cumulative sum of scores in the form of squared differences between any choice of uncertainty function (not necessarily probability) and the indicators of possible combinations of conditional events. Later, Lindley [17] extended Definetti's game by replacing the score function by a much more general form than squared difference, but he did retain the aggregation function as arithmetic sum. Lindley showed, depending on the score function f chosen, a unique corresponding nondecreasing function P_f over unit interval [0,1] back to itself exists, i.e., $P_f:[0,1] \to [0,1]$, such that $P_f(0)=0$, $P_f(1)=1$, and most importantly, if μ is any admissible uncertainty measure (admissibility here must be clarified), then $P_f \circ \mu$ is a finitely additive conditional probability measure. Conversely, if P_f is such that it is strictly increasing over [0,1], then $P_f \circ \mu$ a finitely additive conditional probability measure implies that μ is admissible.

However, Lindley also claimed that a number of well-known uncertainty measures did not satisfy the above basic criteria for being admissible *for any choice of score function f and hence were not admissible in a strong sense.* These included significance tests, Zadeh's max-possibility measure, and Dempster-Shafer belief measures among others. (See [17], pp. 9 et passim.) Despite a lively discussion at the end of Lindley's article by several of the leading researchers in the field, the chief issues involved in the work remained surprisingly untouched:

1. Were Lindley's conclusions concerning the general inadmissibility of the named list of uncertainty measures actually correct?

2. What role does DeFinetti's concept of conditional event indicator functions and their assumed relations play - used tacitly by Lindley and not part of the standard literature (semantic-oriented) for conditional probability ?

3. Can Lindley's almost informal presentation be made more rigorous and put into a pure game theory context for further analysis?

4. What happens when the sum Lindley used for aggrgation is replaced by a more general function? Lindley did address this question in part, but no actual answer was provided.

This led to [18], where it was shown that:

In response to 1: Lindley's conclusion were not entirely true. To begin with, a monotone transform on probability is not the same as probability-unlike other mathematical concepts, probability as a measure is very sensitive to any external transform, being no longer a probability, while for any internal - i.e., within the argument-transform, probability is changed- but to another induced probability! Thus, it is not difficult to show ([18], Theorem 4.2.2) that probability itself is inadmissible for non-square score functions. Moreover, although Lindley correctly concluded that max-possibility measure was not generally admissible for any choice of score function f (still for aggregation as a sum), many related possibility functions are generally admissible and in fact max=possibility can be shown to be a uniform limit, under mild conditions, of generally admissible possibility measures. Furthermore, Lindley's conclusion that all Dempster-Shafer belief measures are not admmisible for any choice of score function is also in error, since the important class of fixed powers of probability measures when the power exceeds or equals unity is generally admissible.

In response to 2: DeFinetti, as well as Lindley, both assumed firstly the validity of the use of conditional event indicator functions as

$$(A|B)(\omega) = \begin{cases} u & (\text{"undetermined"}),\ \text{if } \omega \in B' \\ 1 & ,\ \text{if } \omega \in AB\ \ (\text{i.e., } A \cap B) \\ 0 & ,\ \text{if } \omega \in A'B\ \ (\text{i.e., } B - A) \end{cases} \tag{2.1}$$

for any sets $A,B \subseteq \Omega$, some fixed universal set, $\omega \in \Omega$, and where (A|B), in Von Neumann's spirit, is thought of both as the conditional set or event "A given B" (antecedent B, consequent A) and as its indicator function as given in (2.1). This relation is obviously reasonable. In addition, both Definetti and Lindley assumed the validity of the basic chain intersection relation among conditional events:

$$(AB|C) = (A|BC)\cdot(B|C);\ \text{all } A,B,C \subseteq \Omega. \tag{2.2}$$

This of course corresponds to the well-known conditional probability relation

$$p(AB|C) = p(A|BC)\cdot p(B|C);\ \text{all } A,B,C \in \mathcal{A}, \tag{2.3}$$

for which $p(BC) > 0$, $\mathcal{A} \subseteq P(\Omega)$ boolean algebra of sets, and $p:\mathcal{A} \to [0,1]$ a finitely additive conditional probability measure.

Although DeFinetti briefly mentioned developing a calculus of operations among conditional events, in both volume 1 and 2 of [16] (see [16], vol.1, Chpt. 4 and vol. 2, pp. 266 et passim to 333), he did not take any of these ideas further, nor did Lindley [17]. However, these questions are fully addressed in [19]-[21], where not only is (2.2) derived - as well as (2.1) - from a minimal number of natural assumptions, but also a full calculus of operations and relations is developed in a non ad hoc manner extending the usual ones for the unconditional events describing a typical boolean algebra. In addition, a number of interesting mathematical properties are shown for the ensuing conditional event algebra, including: relations with Koopmann qualicative conditional probability structures; a sound and complete version of propositional CPL (conditional probability logic); and a full algebraic abstraction showing such conditional extensions of boolean algebras are themselves - though not boolean by failure of the Law of Excluded Middle (or Complements) - are indeed bounded distributive lattices which are relatively pseudocomplemented, possess an involution operator, are DeMorgan and absorbing, as well as possessing a number of other properties, and can be shown to extend the Stone Representation Theorem [21]. See also the previous related work of Schay [22] and Calabrese [23].

In response to 3: It is shown in some detail in [18] that bayes decision functions, least favorable prior distributions, game values, and other game theoretic properties can be derived for the DFL uncertainty game. Most importantly, one can not only determine the analytic conditions for admissibility of competing uncertainty measures- as will be outlined later here - but also use directly the overall uncertainty game loss function to rank or evaluate such measures numerically for various given situations. The latter is most compatible with the spirit of developing a comprehensive theory of C^3 systems integrating data fusion.(See sect.6.)

In response to 4: It is shown in [18] that non-sum aggregation functions can be used in determining the overall uncertainty game loss which do not yield probability measures or functions of them as the admissible class. (In particular, see [18], section 7.) This topic will not be considered any further here, except to show some definitions.

In the next section, the rigorous analysis for the DFL uncertainty game is begun.

3. BASIC ESTABLISHMENT OF THE DFL UNCERTAINTY GAME

Let, throughout, Ω be a fixed nonempty set and $\mathcal{A} \subseteq P(\Omega)$ a fixed boolean algebra of subsets of Ω with the usual set notation $\cup$, $\cdot$ (for $\cap$), $(\ )'$, $\emptyset$, etc. Also, let for each positive integer n, $\mathcal{A}^n$ denote the class of all n-sequences or n-tuples of $\mathcal{A}$ with typical element denoted as $\bar{A}=(A_1,\ldots,A_n)$, $A_j \in \mathcal{A}$. In turn, let

$$\mathcal{A}_\infty \stackrel{d}{=} \mathcal{A} \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \cdots \qquad (3.1)$$

be the class of all finite sequences of sets in $\mathcal{A}$.

Further, relative to conditional sets, define:

$$\tilde{\mathcal{A}} \stackrel{d}{=} (\mathcal{A}|\mathcal{A}) \stackrel{d}{=} \{(A|B): A,B \in \mathcal{A}\}, \qquad (3.2)$$

the class of all conditional events of $\mathcal{A}$, and

$$\tilde{\mathcal{A}}_\infty \stackrel{d}{=} (\mathcal{A}|\mathcal{A})_\infty \stackrel{d}{=} \tilde{\mathcal{A}} \cup \tilde{\mathcal{A}}^2 \cup \tilde{\mathcal{A}}^3 \cup \cdots \qquad (3.3)$$

the class of all finite sequences of conditional sets extending $\mathcal{A}$ with typical element denoted as $\alpha = ((A_1|B_1),\ldots,(A_n|B_n))$. Also, for each $B \in \mathcal{A}$, denote the class of all conditional events with antecedent B as

$$\mathcal{A}_B \stackrel{d}{=} \{(A|B): A \in \mathcal{A}\} \subseteq \tilde{\mathcal{A}} \quad ; \qquad (3.4)$$

and define the class of all finite sequences of events having the same antecedent as

$$\tilde{\mathcal{A}}_0 \stackrel{d}{=} \bigcup_{B\in\mathcal{A}} \bigcup_{n=1}^{\infty} (\mathcal{A}|B)^n \subseteq \tilde{\mathcal{A}}_\infty . \qquad (3.5)$$

At this point, consider again DeFinetti's concept of conditional indicator function as given here in (2.1). It is obvious from the definition that

$$(A|\Omega) = A \quad \text{and} \quad (A|B) = (AB|B); \text{all } A,B \in \mathcal{A}. \qquad (3.6)$$

One could also make the following assumptions which are natural homomorphisms relative to consequences for a fixed common antecedent:

ASSUMPTION I:

For all $A,B,C \in \mathcal{A}$,

$$(A|B)\cdot(C|B) = (AC|B) , \qquad (3.7)$$

$$(A|B)\cup(C|B) = (A\cup C|B), \qquad (3.8)$$

$$(A|B)' = (A'|B) . \qquad (3.9)$$

One could also add or replace Assumption I by the analogue of probability conditional chaining forms mentioned earlier which DeFinetti and Lindley employed in their derivations:

ASSUMPTION II:

For all $A,B,C \in \mathcal{A}$,

$$(AB|C) = (A|BC)\cdot(B|C). \qquad (3.10)$$

Assumptions I and/ or II will be stated explicitly where made. Otherwise, only assume that DeFinetti's conditional indicator set function and its property as in (3.6) holds.

Next, let throughout the analysis $a_2 < a_0 < a_1 < a_3$ be fixed real numbers, and recalling the symbol u as introduced in (2.1), let $f:[a_2,a_3]\times\{0,1,u\} \to \mathbb{R}$, where $\mathbb{R}$ denotes the real euclidean line of numbers, be such that:

(i) For each $j \in \{0,1\}$, $f(\cdot,j)$ is continuously differentiable with a unique global minimum in $[a_2,a_3]$ at a_j, so it is strictly increasing over $[a_j,a_3]$ and strictly decreasing over $[a_2,a_j]$.

(ii) $f(\cdot,0) = 0$ over $[a_2,a_3]$.

If f satisfies all of the above, call f a *score function.*

Next, denoting the set of all real finite sequences as

$$\mathbb{R}_\infty \stackrel{d}{=} \mathbb{R} \cup \mathbb{R}^2 \cup \mathbb{R}^3 \cup \cdots , \qquad (3.11)$$

suppose that $\psi:\mathbb{R}_\infty \to \mathbb{R}$ is a function such that:

(a) ψ is continuously differentiable in all of its arguments.

(b) ψ is strictly increasing in each of its arguments seperately over any $\mathbb{R}^n$.

(c) For any positive integer n, letting 0_n be the n by 1 zero vector in $\mathbb{R}^n$,

$$\psi(0_n) = 0 . \qquad (3.12)$$

In this case, call any such ψ an *aggregation function*. The most important aggregation function- and the one that will be used almost entirely here is $\psi = +$, ordinary arithmetic sum.

Next, define

$$\Lambda_{(1)} \overset{d}{=} \tilde{A}_\infty \times \Omega \tag{3.13}$$

and call it the *space of moves or purestrategies of player 1 or Nature*. $\Lambda_{(1)}$ corresponds to all possible values of unknown parameter $\omega \in \Omega$ in combination with all possible available experiments, i.e., finite sequences a of unconditional, or more generally, conditional sets from $\tilde{A}$, with evaluation using (2.1),

$$a(\omega) \overset{d}{=} ((A_1|B_1)(\omega),\ldots,(A_n|B_n)(\omega)) \tag{3.14}$$

thus representing in reality which conditional events $(A_j|B_j)$ in the sequence occur, i.e., for which $\omega \in A_jB_j$, yielding $(A_j|B_j)(\omega)=1$; which do not occur, i.e., for which $\omega \in A_j'\cdot B_j$, yielding $(A_j|B_j)(\omega) = 0$; and which are uncertain in their occurrence, i.e., for which $\omega \in B_j'$, yielding $(A_j|B_j)(\omega) = u$.

Next, define the *canonical partition mapping* $\Pi:A_\infty \to PP(A)$, where for any positive integer n and any $\bar{A} = (A_1,\ldots,A_n) \in A_\infty$,

$$\Pi(\bar{A}) \overset{d}{=} \{A_K: K \subseteq J_n\}, \tag{3.15}$$

where for each subset K,

$$A_K \overset{d}{=} \bigcap_{j\in K} A_j \cap \bigcap_{i\in J_n - K} A_i', \tag{3.16}$$

$$J_n \overset{d}{=} \{1,\ldots,n\}. \tag{3.17}$$

Define also the mapping $\tau:\tilde{A} \to A^3$, where for any $A_j,B_j \in A$,

$$\tau(A_j|B_j) \overset{d}{=} (A_jB_j,\ A_j'B_j,\ B_j'). \tag{3.18}$$

Then, extend τ to $\tau: \tilde{A}_\infty \to \tilde{A}_\infty$, where now for any $a = ((A_1|B_1),\ldots,(A_n|B_n)) \in \tilde{A}_\infty$,

$$\tau(a) \overset{d}{=} (\tau(A_1|B_1),\ldots,\tau(A_n|B_n)), \tag{3.19}$$

identified in the natural way as an element in A^{3n}.

Thus, using (3.19), one can extend Π in (3.15) to $\Pi:\tilde{A}_\infty \to PP(A)$ where now, for any $a \in \tilde{A}_\infty$,

$$\Pi(a) \overset{d}{=} \Pi(\tau(a)). \tag{3.20}$$

Just as $\Pi(\bar{A})$ is a finite disjoint exhaustive partitioning of Ω which is the smallest possible such class from which all elements of $\bar{A}$ can be obtained as disjoint unions of this class, so is $\Pi(a)$ as a disjoint exhaustive partitioning of Ω relative to a, using (2.1).

Hence, one can conclude from above that a considered as a function $a:\Omega \to \{0,1,u\}^n$ can be naturally identified with the restriction

$$a: \Pi(a) \to \{0,1,u\}, \tag{3.21}$$

where typically for any $\gamma \in \Pi(a)$,

$$a(\gamma) \overset{d}{=} a(\omega), \text{ all } \omega \in \gamma, \tag{3.22}$$

is the same constant value depending only on γ.

Next, call

$$\Lambda_2 \overset{d}{=} [a_2,a_3]^{\tilde{A}} \tag{3.23}$$

the *space of moves or pure strategies of player 2 or the Decision-maker*. Equivalently, Λ_2 is the *space of all uncertainty measures* $\mu:A \to [a_2,a_3]$.

In turn, define the *overall loss function* $L_{f,\psi}: \Lambda_{(1)} \times \Lambda_2 \to \mathbb{R}$, where for any $a \in \tilde{A}_\infty$, any $\omega \in \Omega$, and any $\mu \in \Lambda_2$,

$$L_{f,\psi}(a,\omega;\mu) \overset{d}{=} \psi(f(\mu(a),a(\omega))), \tag{3.24}$$

where

$$\mu(a) \overset{d}{=} (\mu(A_1|B_1),\ldots,\mu(A_n|B_n)) \tag{3.25}$$

and

$$f(\mu(a),a(\omega)) \overset{d}{=} (f(\mu(A_j|B_j),(A_j|B_j)(\omega)))_{j\in J_n} \in \mathbb{R}^n. \tag{3.26}$$

But, in view of the identification in (3.21), one can replace $\Lambda_{(1)}$ by Λ_1 and redefine $L_{f,\psi}$ equivalently to (3.24) as $L_{f,\psi}: \Lambda_1 \times \Lambda_2 \to \mathbb{R}$, where

$$\Lambda_1 \overset{d}{=} \{(a,\Pi(a)): a \in \tilde{A}_\infty, \text{ with (3.21) holding}\}. \tag{3.27}$$

Thus, for any $a \in \tilde{A}_\infty$, $\gamma \in \Pi(a)$, $\mu \in \Lambda_2$, one can replace (3.24) by the equivalent

$$L_{f,\psi}(a,\gamma;\mu) \overset{d}{=} \psi(f(\mu(a),a(\gamma))). \tag{3.28}$$

Call

$$G_{f,\psi} \overset{d}{=} (\Lambda_{(1)},\ \Lambda_2;\ L_{f,\psi}) \tag{3.29}$$

the *DeFinetti-Lindley uncertainty measure game (DFL)*. Clearly, $(\Lambda_1,\Lambda_2;L_{f,\psi})$ is equivalent to $G_{f,\psi}$.

Most importantly, it follows that for each $a \in \tilde{A}_\infty$, the subgame

$$G_{a,f,\psi} \overset{d}{=} (\{a\}\times\Omega,\ \Lambda_2;L_{f,\psi}) \tag{3.30}$$

where $L_{f,\psi}$ is restricted appropriately, has player 1's space $\{a\}\times\Omega$ being infinite, while for the equivalent subgame $(a,\Pi(a),\Lambda_2;L_{f,\psi})$, player 1's space $(a,\Pi(a))$ is finite, resulting in an S-game. Such games allow for elegant theoretical results yielding least favorable priors, bayes decision functions all existing, closed and bounded conditional loss (risk) set with a continuous loss function, completeness of all admissible decisions, etc. (See e.g. [24]; see also [18], section 4.1.)

Future work will consider in more detail various game theoretic properties of DFL.

4. ADMISSIBILITY CONCEPTS FOR DFL

Among all the many possible game theoretic properties one could consider for DFL, admissibility and bayes decisions, i.e., bayes uncertainty measures, stand out in importance.

Let $\mu \in \Lambda_2$ be any uncertainty measure and $E \subseteq \tilde{A}_\infty$ arbitrary. Then, define:

(i) μ is *E-admissible with respect to* $G_{f,\psi}$ iff for each $a \in E$, there is *no* $\nu = \nu_a \in \Lambda_2$ (restricted to a without loss of generality) such that

$$L_{f,\psi}(a, \cdot, \nu) \prec L_{f,\psi}(a, \cdot, \mu), \quad (4.1)$$

i.e.,

$$L_{f,\psi}(a,\gamma;\nu) \leq L_{f,\psi}(a,\gamma,\mu);\text{all } \gamma\in\Pi(a), \quad (4.2)$$

with strict inequality holding at least for some γ.

(ii) μ is *E-weak locally admissible wrt* $G_{f,\psi}$ (*E-WLAD*) iff for any $a=((A_1|B_1),\ldots,(A_n|B_n)) \in E$, for each $y \in \mathbb{R}^n$, with $\|y\|=1$ and each $\lambda>0$, there is a positive real number $r=r(\mu,y,\lambda)$ such that there is no $t, 0<t<r$, such that assuming wlog $L_{f,\psi}(a,\gamma,\cdot)$ over $[a_0,a_1]^n$ is non-constant, for all $\gamma \in \Pi(a)$,

$$L_{f,\psi}(a,\gamma,\mu+(ty))-L_{f,\psi}(a,\gamma,\mu)\leq-\lambda t\cdot 1_n, \quad (4.3)$$

where 1_n is the n by 1 vector of all 1's.

(iii) For any $a \in \tilde{A}^n$, the *jacobian matrix* here is

$$J_{f,\psi}(a)(x) \overset{d}{=} (\partial L_{f,\psi}(a,\gamma,x)/\partial x)_{\gamma\in\Pi(a)}, \quad (4.4)$$

an m by n matrix function of $x \in [a_0,a_1]^n$, where $m \overset{d}{=} \text{card}(\Pi(a))$. Then, it is readily shown:

μ is E-WLAD wrt $G_{f,\psi}$ iff for each $a \in E$, there is no $x = x_a \in [a_0,a_1]^n$ such that

$$J_{f,\psi}(a)(x)\cdot x < 0_m. \quad (4.5)$$

(See [18], Theorem 3.2.1.)

(iv) μ is E-bayes *wrt mixed extension of* $G_{f,\psi}$ iff for each $a \in E$, there exists a prior probability function $q=q_a$ (over $\Pi(a)$ such that

$$\inf_{(\nu\in\Lambda_2)}\Big(\sum_{\gamma\in\Pi(A|B)} L_{f,\psi}(a,\gamma,\nu)\cdot q(\gamma)\Big) \quad (4.6)$$

occurs for $\nu = \mu$.

(v) Define the following classes:

$$E_1 \overset{d}{=} \{((A|B)): (A|B) \in \tilde{A}\}, \quad (4.7)$$

$$E_2 \overset{d}{=} \{((A|B),(A'|B)):A,B \in A\}, \quad (4.8)$$

$$E_3 \overset{d}{=} \{((A|B),(C|B),(A\cup C|B)):A,B,C \in A;AC=\emptyset\}, \quad (4.9)$$

$$E_4 \overset{d}{=} \{((A|BC),(B|C),(AB|C)):A,B,C \in A\}. \quad (4.10)$$

Thus,

$$E_1,E_2,E_3 \subseteq \tilde{A}_0; \quad E_4 \subseteq \tilde{A}_\infty. \quad (4.11)$$

(vi) Define for each $t \in [0,1]$,

$$P_f(t) \overset{d}{=} f'(t,0)/(f'(t,0)-f'(t,1)). \quad (4.12)$$

Thus, $P_f:[0,1] \to [0,1]$ is continuous nondecreasing with $P_f(0) = 0$ and $P_f(1) = 1$.

<u>Theorem</u> 4.1. ([18], Corollary 3.2.1)
$\mu \in \Lambda_2$ is E_1-WLAD wrt $G_{f,\psi}$ iff range$(\mu)\subseteq [a_0,a_1]$. ∎

<u>Theorem</u> 4.2. ([18], Theorem 3.2.2)

Suppose Assumption I holds, then

(i) $\mu \in \Lambda_2$ is E_1,E_2,E_3-WLAD wrt $G_{f,+}$

<u>iff</u>

(ii) $P_f\circ\mu : A_B \to [0,1]$ is a finitely additive probability measure, for each $B\varepsilon A$. ∎

<u>Theorem</u> 4.3. ([18], Theorem 3.2.3)

Suppose both Assumptions I and II hold. Then:

(i) $\mu \in \Lambda_2$ is E_1,E_2,E_3,E_4-WLAD wrt $G_{f,+}$

<u>iff</u>

(ii) $P_f\circ\mu : \tilde{A} \to [0,1]$ is a finitely additive *conditional* probability measure, i.e., necessarily, for any $(A|B) \in \tilde{A}$, provided $\mu(B) > 0$,

$$(P_f\circ\mu)((A|B)) = (P_f\circ\mu)(A|B)=(P_f\circ\mu)(AB)/(P_f\circ\mu)(B), \quad (4.13)$$

where of course $P_f\circ\mu\ (B) = P_f(\mu(B))$, etc. ∎

It should be remarked that the condition (ii) of Theorem 4.2 is in general weaker than (ii) for Theorem 4.3. Indeed, Aczél [25], pp. 321-324 has shown in effect a similar result: that the function in question has not only similar properties to $P_f\circ\mu$, but is also, relative to all conditional events, a function of the consequent (conjoined with the antecedent) and the antecedent.

<u>Theorem</u> 4.4. ([18], Theorem 4.2.1)

Suppose Assumption I holds and score function f is such that P_f is strictly increasing over [0,1]. Then, the following statements are equivalent for any given $\mu \in \Lambda_2$ wrt $G_{f,+}$:

(i) μ is $\tilde{A}_0$-admissible.

(ii) μ is E_1,E_2,E_3-WLAD.

(iii) μ is $\tilde{A}_0$-bayes.

(iv) $P_f\circ\mu:A_B \to [0,1]$ is a finitely additive probability measure, for each $B \varepsilon A$.

∎

Theorem 4.5. ([18], Theorem 4.2.1')

Suppose both Assumptions I and II hold and score function f is such that P_f is strictly increasing over [0,1]. Then the following statements are equivalent for any given $\mu \varepsilon \Lambda_2$:

(i) μ is $\tilde{A}_\infty$-admissible.

(ii) μ is E_1,E_2,E_3,E_4-WLAD.

(iii) μ is $\tilde{A}_\infty$-bayes.

(iv) $P_f \circ \mu : \tilde{A} \to [0,1]$ is a finitely additive conditional probability measure.

∎

5. ADMISSIBILITY OF POSSIBILITY AND BELIEF MEASURES

Consider first another concept related to admissibility, noting that the space of all uncertainty measures Λ_2 does not depend on any choice of score function - nor on any aggregate function. However, since throughout this section only the case $\psi = +$ will be treated:

Let $\mu \varepsilon \Lambda_2$ be arbitrary. Then, if there exists a score function f such that μ is $\tilde{A}_0$-admissible wrt game $G_{f,+}$ such that $P_f:[0,1] \to [0,1]$ is strictly increasing, call μ *generally admissible*. If, more strongly, there is a score function with P_f strictly increasing over [0,1] such that μ is $\tilde{A}_\infty$-admissible, call μ *strong generally admissible*. If, more weakly, there is such an f as above so that μ is A_∞-admissible, call μ *weak generally admissible*.

Note: all strictly increasing P_f for appropriate score functions f coincides with class $\mathcal{H}$ of all strictly increasing $h:[0,1] \to [0,1]$ with $h(0)=0$, $h(1)=1$:

Theorem 5.1.

Let $\mu \varepsilon \Lambda_2$ be arbitrary. Then:

(i)

(I) μ is generally admissible

iff

(II) There exists $h \varepsilon \mathcal{H}$ such that $h \circ \mu$ is a finitely additive probability measure over each A_B, $B \varepsilon A$.

(ii) (I) μ is strong generally admissible

iff

(II) There exists $h \varepsilon \mathcal{H}$ such that $h \circ \mu$ is a finitely additive conditional probability over $\tilde{A}$.

(iii)

(I) μ is weak generally admissible

iff

(II) There exists $h \varepsilon \mathcal{H}$ such that $h \circ \mu$ is a finitely additive probability measure over A.

Proofs: Use Theorems 4.4, 4.5 and the above comment concerning strictly increasing P_f's and class $\mathcal{H}$.

∎

Add on the phrase "*countably additively*" to any of the three types of general admissibility, when the f yielding P_f (or euivalently, the $h \varepsilon \mathcal{H}$) is such that $P_f \circ \mu$ is not only a finitely additive probability measure (or similarly, for $h \circ \mu$), but it is countably additive.

Some additional definitions will be required in order to show the general admissibility of a large class of possibility measures:

Call a function $T:[0,1]^2 \to [0,1]$ a *t-conorm* if T is associative, commutative, non-decreasing such that

$$T(s,0) = s \ ; \ T(s,1) = 1 \ ; \text{ all } s\varepsilon[0,1]. \quad (5.1)$$

(For background on this and related concepts discussed below, see e.g. [26].) Call a t-conorm *archimedean*, if it is also continuous with

$$T(s,s) > s \ , \text{ for all } 0<s<1. \quad (5.2)$$

Max is a t-conorm which is not archimedean, but minsum and probsum are archimedean t-conorms, where

$$\text{minsum}(s,t) \overset{d}{=} \min(\text{sum}(s,t),1), \quad (5.3)$$

$$\text{probsum}(s,t) \overset{d}{=} 1-((1-s)\cdot(1-t)), \quad (5.4)$$

$$\text{sum}(s,t) \overset{d}{=} s+t \ . \quad (5.5)$$

A t-conorm can always be extended to $T:[0,1]_\infty \to [0,1]$ where

$$[0,1]_\infty \overset{d}{=} [0,1] \cup [0,1]^2 \cup [0,1]^3 \cup \cdots , \quad (5.6)$$

the set of all finite sequences in [0,1], by first defining for all $s\varepsilon[0,1]$,

$$T(s) \overset{d}{=} id(s) \overset{d}{=} s \ , \quad (5.7)$$

and for all $s_1,s_2,\cdots,s_n \varepsilon [0,1]$,

$$T(s_1,s_2,\cdots,s_n) \overset{d}{=} T(T(s_1,s_2,\cdots,s_{n-1}),s_n), \quad (5.8)$$

using associativity and commutativity.

Ling's Theorem is also relevant here:

Theorem 5.2. (Ling[27])

(i) Let T be any archimedean t-conorm. Then, there exists a function $g=g_T$ called the *generator of T*, such that

(I) $g:[0,1] \to [0,+\infty]$ is strictly increasing, continuous with $g(0)=0$ and $g(1) \le +\infty$.

(II)

$$T(r,s) = g^{-1}(\min(g(r)+g(s),g(1))), \quad (5.9)$$

for all $r,s \varepsilon [0,1]$

(ii) If g is any function satisfying (I) and T is any function defined through g via eq.(5.9), then T is an archimedean t-conorm.

∎

It follows immediately from Ling's Theorem above that if T is any t-conorm with generator g, then for all $s_1,s_2,\cdots,s_n \in [0,1]$, $n=1,2,3,\cdots$,

$$T(s_1,s_2,\cdots,s_n) = g^{-1}(\min(g(s_1)+\cdots+g(s_n),g(1))), \qquad (5.10)$$

which, turn, can be extended to an at most countably infinite number of arguments, by a straightforward continuity limit approach.

Next, if $\mu \in [0,1]^A$, call μ a *decomposable measure* if there exists a function $T:[0,1]^2 \to [0,1]$ called the *composition law of* μ - not necessarily a t-conorm - such that

$$\mu(A\cup B) = T(\mu(A),\mu(B)) \; ; \; A,B \in A; \; AB = \emptyset. \qquad (5.11)$$

Next, if $\mu \in [0,1]^A$ and there exists a t-conorm T such that μ is decomposable wrt T, i.e., T is the compositional law for μ, then say that μ is a *T-possibility measure*. Thus, if $\mu \in [0,1]^A$ is a T-possibility measure, it follows that for any at most countable set $A \in A$, since trivially,

$$A = \underset{\omega\in A}{\cup}\{\omega\} \text{ disjointly}, \qquad (5.12)$$

then

$$\mu(A) = T((\mu(\{\omega\}))_{\omega\in A}). \qquad (5.13)$$

Conversely, if $\mu:\Omega \to [0,1]$ is arbitrary, i.e., a fuzzy set membership function, and if T is any t-conorm, then μ can be extended to $\mu_T:P(\Omega) \to [0,1]$, where for any $A \in P(\Omega)$,

$$\mu_T(A) \overset{d}{=} T((\mu(\omega))_{\omega\in A}) \; , \qquad (5.14)$$

noting immediately that μ_T is a T-possibility measure. Note that any prob. meas. μ is a T-possibility measure with T = minsum.

With all of the above definitions made, the following obtains:

Theorem 5.3. ([18], Theorem 5.2.1 and ensuing remarks)

Suppose that Ω is at most countably infinite and consider only $A = P(\Omega)$, with $\mu \in [0,1]^{P(\Omega)}$. Thus here all concepts of general admissibility coincide with the weak one. Then:

(i)

(I) μ is general admissible

iff

(II) μ is a T-possibility measure, where T is an archimedean t-conorm with generator g such that

$$g(1)=1 \; ; \; \sum_{\omega\in\Omega} g(\mu(\omega)) \le 1. \qquad (5.15)$$

(ii) (I) μ is general admissible countably additively

iff

(II) μ is a T-possibility measure such that T is an archimedean t-conorm with generator g so that

$$g(1)=1 \; ; \; \sum_{\omega\in\Omega} g(\mu(\omega)) = 1. \qquad (5.16)$$

Proof: In the above, note that one has the relation between the score function f making μ A_∞-admissible and generator g:

$$g = P_f \; . \qquad (5.17)$$

∎

Theorem 5.4. ([18], Theorem 5.2.2)

Suppose again Ω is at most countably infinite and consider only $A = P(\Omega)$, etc. as in Theorem 5.3, but now suppose that $\mu:\Omega \to [0,1]$ is any given fuzzy set membership function. Then:

(i) Suppose μ is normalized, i.e.,

$$\mu(\omega_0) = 1 \; ; \text{ some } \omega_0 \in \Omega \; . \qquad (5.18)$$

Then,

(I) There exists T-possibility measure extending μ to $\mu_T:P(\Omega) \to [0,1]$ such that μ_T is general admissible countably additively

iff

(II)

$$\mu = \delta_{\omega_0}, \text{ (Kronecker dlta)} \qquad (5.19)$$

(ii) Suppose

$$0 < x_0 \overset{d}{=} \sup_{\omega\in\Omega} \mu(\omega) < 1 \; . \qquad (5.20)$$

Then,

(I) holds as in (i) but with (5.18) replaced by (5.20)

iff

(II) $\mu^{-1}[t,1]$ is a finite set for any $0 < t \le 1$

Proof: The proof of (i) is simple. The proof of (ii) (I) implies (II) is also simple and not of any significance, but the proof of (II) implies (I) is complicated. The latter is very significant in that it provides a general constructive way for obtaining a generator g, which in turn determines an appropriate t-conorm T_g; where g satisfies (5.16), and hence by the proof of Theorem 5.3, score function f making μ_{T_g} A_∞-admissible is determined as in (5.17).

Unfortunately, due to lack of space, the interested reader is referred to [18], Theorem 5.2.2 for full details of the long construction. ∎

It should be remarked that the significance of Theorem 5.4 is that it allows essentially any fuzzy set membership function over a discrete domain to be extended appropriately to an admissible uncertainty measure over the power class. Even if the fuzzy set membership function is normalized - as is often the case - by simply establishing a slight deficiency, i.e. a maximum less than unity as in (5.20), the adjusted function can then be used as above to lead to an admissible extension! Finally, note that the condition in (ii)(II) is satisfied for all $\mu:\Omega \to [0,1]$, when Ω is finite.

Theorem 5.5. ([18], Theorem 5.2.3)

Suppose the same general hypothesis holds here as in Theorem 5.4 (ii). Suppose also, that there exist nonnegative real numbers t_1,t_2 such that

$$\mathrm{card}(\mu^{-1}[1/n \; , \; 1/(n-1)]) \le t_1 \cdot n^{t_2} \; , \qquad (5.21)$$

for all positive integers $n \ge 2$.

Then, though it is easily seen that the t-conorm extension in the form of Zadeh's max possibility measure μ_{max} is not general admissible (see Theorem 5.3, noting as before that max is not an archimedean t-conorm), it is the uniform limit in all $A \in P(\Omega)$ such that A is disjoint from the set $\mu^{-1}(x_0)$, card(A) bounded, of the general admissible T_r-possibility measure extensions $\mu_{\overline{T}}$ of μ, as r approaches $+\infty$, where for all $s_j \in^r [0,1]$, $j=1,\ldots,n$,

$$T_r(s_1,\ldots,s_n) \stackrel{d}{=} \min(s_1{}^r+\cdots+s_n{}^r,1)^{1/r}. \quad (5.22)$$

∎

This section is concluded with a brief result concerning Dempster-Shafer belief measures. For more extensive treatment, see again [18]. For definitions and background, including relations between random set supercoverages and Poincaré expansion generalizations, see [25].

Apropos to Lindley's conclusion that essentially all belief measures are not generally admissible:

Theorem 5.6. ([18], Theorem 6.2)

Suppose that Ω is a finite space and once more consider only $A = P(\Omega)$ with A_1 in effect restricted to $[0,1]^A$. Also, consider the class

$$\mathcal{B} \stackrel{d}{=} \{\mu^r : \mu:A \to [0,1] \text{ is a finitely additive probability measure, } r \geq 1 \text{ arbitary real}\} \quad (5.23)$$

Then, obviously by Theorem 5.1 (iii) with $h(\ ) = (\ )^r$, $\mathcal{B}$ is a class of (weak) generally admissible uncertainty measures. But, in addition, $\mathcal{B}$ is a class of Dempster-Shafer belief measures.

Proof: See [18], where the criterion for belief functions is verified for $\nu = \mu^r$:

$$\sum_{B \subseteq A} (-1)^{card(A-B)} \cdot \nu(B) \geq 0 \ ; \text{all } A \in A. \quad (5.24)$$

∎

6. FURTHER INTERPRETATION OF DFL GAME AND USE WITH C^3 SYSTEM MODEL

The thrust of sections 2-5 was to show that the DFL game - on at least a theoretical level - provides criteria for determining the admissibility or bayesian status for any given uncertainty measure: namely it must be essentially a monotone increasing transform of a probability measure. However, these results, in themselves, are not enough to provide practical guidelines for direct comparisons of uncertainty measures. One must be able to compare losses directly and numerically.

Assume throughout most of the following analysis- unless otherwise indicated- that $\phi = +$, Ω is a nonvacuous finite space, and restrict A_2 so that only unconditional events are considered, i.e., $A_2 = [0,1]^A$, $A = P(\Omega)$. For any $\bar{A}=(A_1,\ldots,A_n) \in A_\infty$, $n \geq 1$, canonical partitioning $\Pi(\bar{A})=(\gamma_1,\ldots,\gamma_m)$, say, with γ_i all nonvacuous disjoint, and for any uncertainty measure $\mu:A \to [0,1]$, note: In regard to (4.6), one can write, for any *known prior* probability measure $q:A \to [0,1]$, the *expected loss wrt q* as

$$\rho_{f,+}(\bar{A},\mu;q) \stackrel{d}{=} \int_{\omega\in\Omega} L_{f,+}(\bar{A},\omega,\mu)\, dq(\omega)$$
$$= \sum_{i=1}^{m} L_{f,+}(\bar{A},\gamma_i,\mu)\cdot q(\gamma_i)$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{m} f(\mu(A_j),A_j(\gamma_i))\cdot q(\gamma_i)$$
$$= \sum_{j=1}^{n} \left(f(\mu(A_j),1)\cdot q(A_j) + f(\mu(A_j),0)\cdot(1-q(A_j)) \right). \quad (6.1)$$

Hence, (6.1) shows (see definition 4(iv) and eq. (4.6)) that the minimal expected loss is

$$\inf_{\substack{\text{over all} \\ \mu \in A_2}} \left(\rho_{f,+}(\bar{A},\mu;q)\right) = \rho_{f,+}(\bar{A},\gamma,\mu_q;q) \ , \quad (6.2)$$

where μ_q is $\{\bar{A}\}$-bayes wrt mixed extension game of $G_{f,+}$ and is given by differentiating (6.1) wrt $\mu(A_j)$:

$$f'(\mu(A_j),1)q(A_j) + f'(\mu(A_j),0)(1-q(A_j)) = 0 \ ,$$

which is equivalent to

$$P_f(\mu_q(A_j)) = q(A_j) \ , \ j=1,\ldots,n, \quad (6.3)$$

i.e.,

$$P_f \circ \mu_q = q \ . \quad (6.4)$$

Also, as q varies arbitrary, compatible with Theorem 4.4,

$$\{ \mu_q : q:A \to [0,1] \text{ is any finitely additive prior probability measure}\}$$
$$= \text{set of all } A_\infty\text{- admissible uncertainty measures.} \quad (6.5)$$

In particular, consider DeFinetti's original square loss function $f = f_0$, where

$$f_0(t,j) \stackrel{d}{=} c_0\cdot(t-j)^2;\ t\in[0,1];j=1,2 \ ; \quad (6.5)$$

c_0 a fixed positive constant, which through further study is negligable. Then, not only are all of the above results valid for this case, but reduce to:

$$P_f(t) = 2c_0t/(2c_0t-2c_0(t-1)) = t \ ;\ t\in[0,1], \text{i.e.}$$

$$P_f = \text{identity function}; \quad (6.6)$$

so that for any prior $q:A \to [0,1]$,

$$\mu_q = q \quad (6.7)$$

with minimal expected loss

$$\rho_{f_0,+}(\bar{A},\mu_q;q) = c_0 \sum_{j=1}^{n} \left((1-q(A_j))^2 q(A_j) + \right.$$

$$q(A_j)^2(1-q(A_j))\Big) \; ;$$

so that simplifying,

$$\rho_{f_0,+}(\bar{A},\mu_q;q) = c_0 \sum_{j=1}^{n} ((1-q(A_j))q(A_j)). \quad (6.8)$$

Moreover, if $\mu \in \Lambda_2$ is arbitrary, then by adding and subtracting $q(A_j)$ from $\mu(A_j)$ inside (6.1) and noting the sum of cross terms is a sum of zero,

$$\rho_{f_0,+}(\bar{A},\mu;q) = \rho_{f_0,+}(\bar{A},\mu_q;q) + \mathcal{z}(\bar{A},\mu-q) \; , \quad (6.9)$$

where

$$\mathcal{z}(\bar{A},\mu-q) \overset{d}{=} c_0 \cdot \sum_{j=1}^{n} (\mu(A_j)-q(A_j))^2. \quad (6.10)$$

Hence, if score function f_0 is used and prior probability measure $q:A \rightarrow [0,1]$ is known, then a natural way of evaluating any competing uncertainty measure is to use $\mathcal{z}(\bar{A},\mu-q)$, whose minimum is of course zero for $\mu = \mu_q = q$.

On the other hand, if q is not known, one could seek e.g., the *least favorable prior* q_0, i.e., that q maximizing (6.2) in general, and for $f=f_0$, (6.8). Thus, for $f=f_0$, n=2, and $\bar{A} = (A_1,A_2)$ with, say,

$$\gamma_1 \overset{d}{=} A_1A_2 \; ; \; \gamma_2 \overset{d}{=} A_1'A_2 ; \; \gamma_3 = A_1A_2' \; ; \; \gamma_4 \overset{d}{=} A_1'A_2' \; , \quad (6.11)$$

with all $\gamma_i \neq \emptyset, \Omega$; so that

$$A_1 = \gamma_1 \cup \gamma_3 \quad ; \; A_2 = \gamma_1 \cup \gamma_2 \quad (6.12)$$

and

$$q(A_1) = q(\gamma_1)+q(\gamma_3); \; q(A_2)=q(\gamma_1)+q(\gamma_2). \quad (6.13)$$

Thus, (6.13) shows (6.8) becomes

$$\rho_{f_0}(\bar{A},\mu_q;q) = c_0 \cdot ((1-q(A_1))q(A_1) + (1-q(A_2))q(A_2))$$
$$= c_0 \cdot \big(2(1-\gamma_4)\gamma_1 + (1-\gamma_3)\gamma_3 + (1-\gamma_2)\gamma_2\big). \quad (6.14)$$

Then, by use of Lagrange multipliers, or by inspection of (6.14), it follows that the least favorable q_0 here is given as

$$q_0(\gamma_2)=q_0(\gamma_3)=x \; ; \; q_0(\gamma_1)=q_0(\gamma_4)=(1/2)-x; \quad (6.15)$$

for any $0 \leq x \leq 1/2$. This, by (6.13), is equivalent to

$$q_0(A_1) = q_0(A_2) = 1/2 \; , \quad (6.15)$$

which is intuitively one would expect.

Another important concept is: for a given uncertainty $\mu \in \Lambda_2$, when prior q is not at all known, determine the best and worst expected losses for all possible prior q. In light of (6.9),(6.10), a reasonable approximation or replacement to this is to consider

$$K(\bar{A},\mu) \overset{d}{=} \inf_{\substack{\text{over all} \\ \text{prior } q}} (\mathcal{z}(\bar{A},\mu-q)). \quad (6.16)$$

Thus, e.g., if $\bar{A}$ is an actual disjoint exhaustive partitioning of nonvacuous sets wrt Ω (so that one can let without loss of generality $A_j = \gamma_j$) and if $\sum_{j=1}^{n} \mu(A_j) \leq 1$, then it is eas· ·o show $K(\bar{A},\mu)$ occurs uniquely for $q=q_\mu$, where , for j=1,...,n ,

$$q_\mu(A_j) = \mu(A_j) + (1/n)(1-\sum_{k=1} \mu(A_k)) \; , \quad (6.17)$$

with corresponding value

$$K(\bar{A},\mu) = (1/n)(1-\sum_{k=1}^{n} \mu(A_k))^2 \; , \quad (6.18)$$

which obviously approaches zero as n approaches $+\infty$. Hence, relative to the criterion in (6.16), all $\mu \in \Lambda_2$ for the above assumptions behave asymptotically as if they were the optimal solution q.

Consider again Theorem 5.4(ii) with Ω finite and as above $\Lambda_2 = [0,1]^A$, $A=P(\Omega)$. Then, compatible with the last remark following the theorem, it is clear by a continuity argument, there is a unique $r=r_\mu$, for each given $\mu:\Omega \rightarrow [0,1]$, such that

$$\sum_{\omega\in\Omega} \mu(\omega)^r = 1 \; . \quad (6.19)$$

Hence, by Theorem 5.3(ii), one can choose $g()=()^r$, and in turn determine archimedean t-conorm T_r (see (5.22)) which makes the extension $\mu_T : A \rightarrow [0,1]$ A_∞-admissible wrt game $G_{f_r},+$, and

$$P_{f_r}(\;) = g(\;) = (\;)^r \; . \quad (6.20)$$

It is also easily verified that one can choose for f_r,

$$f_r(t,0)=t^{r+1} \; ; \; f_r(t,1) = t^{r+1}-(r+1)t + r; \quad (6.21)$$

for all $t\in[0,1]$.

Finally, note , using steps 1-8 in section 1 the joint performance sensitivity tradeoff forms:

$$\left.\begin{array}{l} \text{OVERALL } C^3 : \text{LOSS}(C^3)=\mathcal{L}\circ\mathcal{U}\circ\mathcal{M}\circ AV\circ\mathcal{G}(\mu) \\ \text{vs.} \\ \text{LOCAL DFL LOSS:} \qquad \rho_{f,\psi}(\alpha,\mu;q) \end{array}\right\} , \quad (6.22)$$

where $\mu\in\Lambda_2$ is identified with $(P_{ALDP}, \|\cdot\|, \|v\|)$.

SUMMARY

This paper has been written to demonstrate the potential use of the extended DeFinetti-Lindley Uncertainty Measure Game in comparing and contrasting various uncertainty measures, including probability and fuzzy sets/possibility. Since the data fusion aspect is central to all data fusion, choices of Algebraic Logic Description Pairs can play a key role in evaluations. Synthesizing the two by use of the sequence of computations leading to the C^3 Design Game appears to be a reasonable path to take. Thus, one combines the loss function for the DFL game as a function of the candidate uncertainty measure (compared to the optimal bayes, e.g.- which will always be a function of probability) with the overall C^3 evaluation function. Sensitivity to parameter changes can be obtained through the standard use of matrix differentiation as a chained multiplicative function. Of course, because of the great number of inputs - even in the very simplified model proposed here - computation difficulties arise immediately. On the other hand efficient pruning techniques and judicious use of variable relations can reduce computations.

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